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FINITE-LEVEL QUANTIZED SYNCHRONIZATION OF DISCRETE-TIME LINEAR MULTIAGENT SYSTEMS WITH SWITCHING TOPOLOGIES*

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Abstract. In this paper, the synchronization of discrete-time linear multiagent systems is studied with finite communication data rate and switching topology flows. A class of quantized-observer-based communication schemes and a class of certainty-equivalence-principle-based cooperative control laws are proposed with adaptive encoders and decoders. It is shown that if the pairs of agents' state matrices and control matrices multiplied by Laplacian eigenvalues of the weakly connected components are simultaneously stabilizable, and the communication topology flow is frequently connected, then there exist such protocols leading to synchronization exponentially fast. Furthermore, only finite bits of information exchange per step are required to guarantee the synchronization if the communication channels are frequently active. For first-order dynamics, the dwell time and the number of bits are both related to the unstable mode of agent dynamics, the number of agents, the frequency of connectivity, and the Laplacian eigenvalue ratio of the switching topology flow.

Key words. synchronization, linear multiagent system, quantized observer, switching topology flow

AMS subject classifications. 93E03, 93E15, 60H10, 94C15

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1. Introduction.

With the rapid development of information technology, control systems evolve from single-agent systems to large-scale networked multiagent systems. In recent years, the communication and cooperative control of multiagent systems attracts more and more attention of scholars and engineers, for example, the tracking control of unmanned air vehicles and the formation control of robot teams. Many scholars have made great efforts to study the synchronization or consensus problem, which is the most basic issue of the cooperative control. They established fruitful theoretical results and engineering applications successfully [1], [2], [3], [4].

In early studies, it is always assumed that the communication channels have infinite bandwidths and are capable of transmitting real-valued information precisely. However, the bandwidths of communication channels are always finite in real networks. By using digital communication techniques, the information exchange between agents is an integrated progress of encoding, transmitting, and decoding [5]. From

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this point, the study of synchronization problems with digital communication is of both theoretical and practical importance [6], [7], [8], [9].

For the case with fixed topologies, Kashyap, Basar, and Srikant [6] first studied the quantized consensus of multiagent systems. Assuming each agent holds an integer state, the authors designed a "gossip" protocol to drive all agents' states to an integer approximation of the average of their initial values. In [5], Carli, Bullo, and Zampieri designed encoders and decoders based on infinite-level logarithmic quantizers and proved that the system will achieve precise average consensus if the quantization density is sufficiently high. Li et al. [8] proposed a dynamic encoding-decoding scheme with finite-level uniform quantizers and vanishing scaling functions. They proved that the system can achieve average consensus exponentially fast with a single bit information exchange per time step. You and Xie [9] studied the synchronization of linear multiagent systems and proved that if each agent is stabilizable, and the product of its unstable modes is less than an upper bound depending on the structure of the network topology, then there exist proper communication and control protocols to ensure synchronization. In addition to the data-rate constraint, the switching of network topologies due to packet losses, link failures, or high-level commands is another kind of uncertainty in multiagent systems. Li and Xie [10] studied the quantized averaging of integrator systems under bidirectional information communication. They proposed a class of adaptive quantized communication schemes such that each agent adjusts the range of quantization dynamically according to the information communication with neighbors. By algebraic graph theory, Lyapunov method, and the stability of time-varying systems, it is proved that if the network topology flow is jointly connected, then 3 bits of information exchange per time step suffice for the system to achieve average consensus. Zhang and Zhang [11] and Li et al. [12] studied the synchronization of integrators over unidirectional switching networks. Zhang and Zhang [11] proved that if the network topology flow jointly contains spanning trees and is always balanced, then the system achieves average consensus by properly designing the quantization parameters. Li et al. [12] extended the results of [10] and [11] to general unbalanced switching topology flows. They proved that if the network topology flow is jointly strongly connected, each agent sends 3-bits information to its neighbor and 1-bit information to itself at each step, then the system achieves synchronization. Olshevsky [13] considered distributed consensus under ternary information exchange and proved that for a class of time-varying undirected graphs under some connectivity conditions, the proposed protocol can ensure average consensus with a polynomial convergence time. More recent results on consensus with quantized communication can be found in [14], [15], [16], [17], [18], and the references therein.

The above literature on quantized synchronization with switching topologies all focus on integrator dynamics, while for general linear dynamics, most of the existing literature focus on the case with precise communication. In [19], Wang, Cheng, and Hu proved that for continuous-time controllable linear multiagent systems, distributed control protocols can be found to ensure synchronization if the network topology flow is frequently connected with a positive dwell time. For neutrally stable agent dynamics, Su and Huang [20] proved that the synchronization can be achieved if the network topology flow is jointly connected. By using the nonnegative matrix theory, Qin, Gao, and Yu [21] proved that unstable agents with full row rank input matrices can achieve exponential synchronization if the communication topologies jointly have spanning trees frequently. As we know, up to now, there is no study on the synchronization of general linear multisystems with finite communication data rate and switching topology flows.

In this paper, we study the synchronization of discrete-time linear multiagent systems with finite communication data rate and switching topologies. Compared with former works, the problem has the following new challenges: (i) different from the fixed topology case [8], [9], the quantized synchronization with switching topology flows requires each agent to observe its neighbors's states with intermittent information. It means that the state observation scheme should be adaptive to the interruption of information communication caused by the network topology switching. For the communication among agents, the switching of network topologies also leads to information mismatching between the senders and the receivers, so the broadcasting type encoders and decoders proposed in [8] cannot be used here; (ii) different from the case with neutrally stable agent dynamics, if there is an unstable mode in agent dynamics, then the closed-loop system will diverge exponentially during the period that the network topology is not strongly connected. The divergence of synchronization errors in such periods should be well estimated and restrained such that the stability of the closed-loop system is not affected; (iii) the coupling among the above difficulties due to the agent dynamics, switching topology flows, and the quantization nonlinearity makes the closed-loop analysis much more difficult. In this paper, we propose a class of quantized-observer-based dynamic communication schemes and certainty-equivalence-principle-based control laws. To avoid information mismatching between the senders' encoders and the receivers' decoders, we adopt a channel-activeness-based state updating rule for the encoder-decoder design. To avoid the saturation of quantizers which may lead to the growth of quantization errors, we adjust the quantization levels of each quantizer adaptively according to the status of the channels for the last step. To estimate the divergence of the synchronization errors when the network topology is not strongly connected, we use the weakly connected components of the topology flow to regroup the agents and estimate the difference of every two agents' states by calculating innergroup and intergroup distances. We show that if the network topology flow is frequently connected with a large dwell time for those being strongly connected, and there exist $K \in \mathbb{R}^{m\times n}$ such that $A - \lambda_i BK, i = 1, \dots, l$, are stable, where (A, B) is the pair of agents' state and control matrices, $\lambda_i, i = 1 \dots, l$, are nonzero eigenvalues of Lapalacian matrices of all weakly connected components of the topology flow, then there exist communication and control protocols ensuring synchronization exponentially fast. In addition, if the communication channels are frequently active, then finite bits of information exchange per step can ensure the exponential synchronization. Especially, for unstable first-order systems, the lower bound of the dwell time and the upper bound of the number of bits are given in quantitative relation to the unstable mode of agent dynamics, the number of agents, the frequency of connectivity, and the synchronizability of the topology flow, which is defined as the ratio of the smallest and largest nonzero Laplacian eigenvalues of the connected components.

The rest of this paper is organized as follows. In section 2, we give some preliminaries and formulate the problem to be studied. In section 3, we first consider the case of precise communication, and then the case of finite-level quantized communication, and give the main theorems of this paper. In section 4, we verify the effectiveness of the theoretical results by numerical simulations. The concluding remarks and some future topics are given in section 5.

The following notations will be used in this paper. Denote the column vector with a proper dimension and all elements being 1 and the matrix with all elements being 0 by $\mathbf{1}$ and $\mathbf{0}$, respectively. We use I_n (or I) to represent an identity matrix with dimension n (or with a proper dimension). Denote the sets of real numbers, conjugate

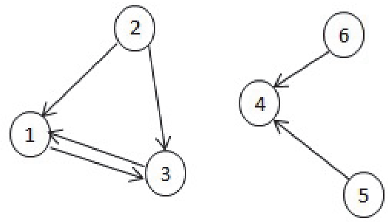

FIG. 1. *Two weakly connected components of a digraph.*

numbers, and nonnegative integers by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$, respectively, and $\mathbb{R}^n$ represents the n-dimensional real space. For a vector $\mathcal{X} \in \mathbb{R}^n$ or a matrix $\mathcal{X} \in \mathbb{R}^{n\times m}$, its transpose is denoted by $\mathcal{X}^H$, its Euclidean norm and the infinite norm are denoted by $\|\mathcal{X}\|$ and $\|\mathcal{X}\|_\infty$, respectively. The spectral radius of a matrix $\mathcal{X}$ is denoted by $\rho(\mathcal{X})$. For two series of numbers $f(k)$ and $g(k)$, we say that $f(k) = O(g(k))$ if there exists a positive constant C such that $|f(k)| \leq C|g(k)|$, $k \in \mathbb{N}$. For a real number r, we use the notation $\lceil r \rceil$ to represent the least integer larger than r. The Kronecker product, denoted by $\otimes$, facilitates the manipulation of matrices by the following properties: (1) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$; (2) $(A \otimes B)^H = A^H \otimes B^H$. Denote the block diagonal matrix with its diagonal matrices being $A_1, \dots, A_k$ by $diag(A_1, \dots, A_k)$. We denote the open ball $\{\mathcal{X} \in \mathbb{R}^n | \|\mathcal{X}\|_\infty < r\}$ by $\mathscr{B}_r$.

2. Preliminaries. In this section, we give some basic concepts in graph theory and formulate the problem to be studied.

2.1. Basic concepts. For a digraph $\mathcal{G} = \{\mathscr{V}, \mathscr{E}\}$, $\mathscr{V} = \{1, \dots, N\}$ is the node set and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the edge set. Denote the adjacency matrix of $\mathcal{G}$ by $\mathscr{A}_\mathcal{G} = [a_{ij}] \in \mathbb{R}^{N\times N}$. If $(j, i) \in \mathscr{E}$, then $a_{ij} = 1$, otherwise $a_{ij} = 0$. Here, we assume $a_{ii} = 0$, $i = 1, \dots, N$. We say that $\mathcal{G}$ is undirected if $\mathscr{A}_\mathcal{G}^H = \mathscr{A}_\mathcal{G}$. Denote the in-degree and out-degree of node i by $deg_i^{in} = \sum_{j=1}^N a_{ij}, deg_i^{out} = \sum_{j=1}^N a_{ji}$, respectively, and denote the degree matrix by $\mathscr{D}_\mathcal{G} = diag(deg_1^{in}, \dots, deg_N^{in})$. We say that $\mathcal{G}$ is balanced if $deg_i^{in} = deg_i^{out}, i = 1, \dots, N$. The Laplacian matrix of $\mathcal{G}$, denoted by $\mathcal{L}_\mathcal{G}$, is defined as $\mathcal{L}_\mathcal{G} = \mathscr{D}_\mathcal{G} - \mathscr{A}_\mathcal{G}$, and its eigenvalues in an ascending order of real parts are denoted by $\lambda_1(\mathcal{L}_\mathcal{G}) = 0, \lambda_i(\mathcal{L}_\mathcal{G}), i = 2, \dots, N$. Denote the Jordan canonical of $\mathcal{L}_\mathcal{G}$ and the associated transformation matrix by $diag(\mathbf{0}, J_2^\mathcal{G}, \dots, J_{k_\mathcal{G}}^\mathcal{G})$ and $\Phi_\mathcal{G}$, respectively, that is, $\Phi_\mathcal{G}\mathcal{L}_\mathcal{G}\Phi_\mathcal{G}^{-1} = diag(\mathbf{0}, J_2^\mathcal{G}, \dots, J_{k_\mathcal{G}}^\mathcal{G})$. A sequence of edges $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)$ is called a path from i_1 to i_k. For a digraph $\mathcal{G} = \{\mathscr{V}, \mathscr{E}\}$, we say that $\mathcal{G}$ is strongly connected if there is a path from each node to any other. An undirected graph $\mathcal{G}$ is called connected if for each pair of nodes, there is a path between them. For a digraph $\mathcal{G} = \{\mathscr{V}, \mathscr{E}\}$, if there is a digraph $\mathcal{G}' = \{\mathscr{V}', \mathscr{E}'\}$ such that $\mathscr{V}' \subseteq \mathscr{V}$ and $\mathscr{E}' \subseteq \mathscr{E}$, then $\mathcal{G}'$ is called a subgraph of $\mathcal{G}$, which is denoted by $\mathcal{G}' \subseteq \mathcal{G}$. We say that $\mathcal{G}' = \{\mathscr{V}', \mathscr{E}'\}$ equals $\mathcal{G}'' = \{\mathscr{V}'', \mathscr{E}''\}$ if $\mathscr{V}' = \mathscr{V}''$ and $\mathscr{E}' = \mathscr{E}''$. A subgraph $\mathcal{G}'$ of the undirected graph $\mathcal{G}$ is called a connected component if (i) $\mathcal{G}'$ is connected, (ii) for any $\mathcal{G}'' \subseteq \mathcal{G}$, if $\mathcal{G}' \subseteq \mathcal{G}''$ and $\mathcal{G}''$ is not equal to $\mathcal{G}'$, then $\mathcal{G}''$ is disconnected. For a digraph $\mathcal{G}$, its mirror graph is an undirected graph, denoted by $\hat{\mathcal{G}} = \{\mathscr{V}, \hat{\mathscr{E}}\}$, where $(j, i) \in \hat{\mathscr{E}}$ if and only if $(i, j) \in \mathscr{E}$ or $(j, i) \in \mathscr{E}$. A subgraph $\mathcal{G}'$ of the digraph $\mathcal{G}$ is called a weakly connected component if $\hat{\mathcal{G}}'$ is a connected component of $\hat{\mathcal{G}}$. For example, the digraph with nodes set $\{1, \dots, 6\}$ in Figure 1 has two weakly connected components.

Given a piecewise constant switching signal $\sigma(t)$: $\mathbb{N} \to \mathbb{N}$, we can define a sequence of switching digraphs $\{\mathcal{G}_{\sigma(t)} = \{\mathscr{V}, \mathscr{E}_{\sigma(t)}\}, t \in \mathbb{N}\}$. Define $0 = t_0 < t_1 < \cdots < t_i <$

$t_{i+1} < \cdots \leq +\infty$ as the switching times of $\sigma(t)$, such that $\mathcal{G}_{\sigma(t)}$ is fixed during each interval $[t_i, t_{i+1})$, $i \in \mathbb{N}$. If there is an integer j such that $\sigma(t)$ is fixed for $t \geq t_j$, then $t_{j+1} = +\infty$. For a sequence of switching graphs $\mathcal{G}_{\sigma(t)}$, where $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots \leq +\infty$ are the switching times, we call $t_{i+1} - t_i$ as the dwell time [19] of graph $\mathcal{G}_{\sigma(i)}$, $i = 1, 2, \ldots$.

2.2. Problem formulation. Consider a discrete-time multiagent system of N agents with the following dynamics,

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i = 1, \ldots, N, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and the control input of agent i. The communication topology flow among agents is modeled by a sequence of switching digraphs $\{\mathcal{G}_{\sigma(t)}, t \in \mathbb{N}\}$, where $\sigma(t)$: $\mathbb{N} \to \{1, \ldots, M\}$. An edge $(j, i) \in \mathscr{E}_{\sigma(t)}$ if and only if the communication channel from agent j to agent i is active at time t; then agent i is called the receiver and agent j is called the sender, or i's neighbor. The set of agent i's neighbors at time t is denoted by $N_i(t) = \{j \in \mathscr{V} | (j, i) \in \mathscr{E}_{\sigma(t)}\}$. Denote $\mathscr{N}_i = \bigcap_{k=1}^{\infty} \bigcup_{t=k}^{\infty} N_i(t)$. We have the following assumptions.

(A1) The eigenvalues of A are not all inside the unit circle.

(A2) The set $\mathscr{N}_i = \bigcup_{t=1}^{\infty} N_i(t)$.

The agent dynamics (2.1) together with the communication topology flow $\{\mathcal{G}_{\sigma(t)}, t \in \mathbb{N}\}$, is called a dynamic network [2], which is denoted by $(A, B, \mathcal{G}_{\sigma(t)})$. Let $\mathscr{G} = \big\{\mathcal{G} | \mathcal{G}$ is a weakly connected component of digraphs in $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_M\}\big\}$. Define Λ as the collections of all nonzero eigenvalues of the Laplacian matrices of the digraphs in $\mathscr{G}$. For the case where $\mathcal{G}_1, \ldots, \mathcal{G}_M$ are undirected graphs, we denote $\lambda_M = \max_{\lambda \in \Lambda} \lambda$ and $\lambda_m = \min_{\lambda \in \Lambda} \lambda$.

For the dynamic network $(A, B, \mathcal{G}_{\sigma(t)})$, a control protocol is called distributed if the control input of each agent depends only on the information of its own and its neighbors. We say that the dynamic network $(A, B, \mathcal{G}_{\sigma(t)})$ achieves synchronization if there is a distributed control protocol such that the closed-loop system satisfies $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, $i, j = 1, \ldots, N$.

Remark 2.1. Note that $\mathscr{N}_i$ denotes the set of agents which are the neighbors of agent i for an infinite number of times. (A2) implies that if a communication channel to agent i is active at some time, then it is active for infinite times. For the asymptotic properties of the dynamic network, the impact of the channels being active for finite times is negligible. For saving resources, we only design communication protocols for the communication channels being active for infinite times, and for the conciseness of the closed-loop analysis, we make Assumption (A2).

In real digital networks, the communication channels have finite bandwidth and the real-valued states are encoded into finite symbols before transmitting. The information communication is an integrated progress of encoding, transmitting, and decoding. Our goal is to study under what conditions does there exist proper communication and control protocols to ensure the dynamic network achieves synchronization with finite communication data rate and a switching network topology flow.

3. Main results. In this section, we give the main results of this paper. First, for the case of precise communication, we present an exponential convergence result under the frequently strongly connected condition. Then based on the exponential convergence result for the case of precise communication, we propose a class of dynamic encoding-decoding communication schemes and a class of certainty-equivalence-principle-based control protocols for the case of finite-level quantized communication.

Conditions will be given on the existence of such a type of communication and control protocols for ensuring all agents achieve synchronization.

3.1. Case of precise communication. We first consider the case of precise communication, where the communication channels are capable of transmitting real-valued information precisely. We have the following assumptions.

(A3) $\mathcal{G}_{\sigma(t)}$ is balanced, $t \in \mathbb{N}$.

(A4) There exists a matrix $K \in \mathbb{R}^{m\times n}$, such that $\max_{\lambda\in\Lambda} \rho(A-\lambda BK) < 1$.

(A5) There is a constant $T \geq 1$ such that for any given time t, there is $t^* \in [t, t+T]$ with $\mathcal{G}_{\sigma(t^*)}$ being strongly connected.

We consider the following class of distributed control protocols:

$$u_i(t) = K \sum_{j\in N_i(t)} a_{ij}(t)(x_j(t) - x_i(t)), \quad i = 1, \ldots, N. \tag{3.1}$$

Let $X(t) = (x_1^H(t), \ldots, x_N^H(t))^H$, $U(t) = (u_1^H(t), \ldots, u_N^H(t))^H, \bar{x}(t) = (\frac{1}{N}\mathbf{1}_N^H \otimes I_n) \cdot X(t), \delta_i(t) = x_i(t) - \bar{x}(t)$ and $\delta(t) = (\delta_1^H(t), \ldots, \delta_N^H(t))^H$, which is called the synchronization error. Denote $C_1 = \max_{\mathcal{G}\in\mathscr{G}} \|\Phi_{\mathcal{G}}\|$ and $C_2 = \max_{\mathcal{G}\in\mathscr{G}} \|\Phi_{\mathcal{G}}^{-1}\|$. For a matrix $\mathcal{X} \in \mathbb{R}^{m\times n}$, denote $A(\mathcal{L}_{\mathcal{G}}, \mathcal{X}) = I \otimes A - diag(J_2^{\mathcal{G}}, \ldots, J_{k_{\mathcal{G}}}^{\mathcal{G}}) \otimes B\mathcal{X}, \rho_\Lambda(\mathcal{X}) = \max_{\mathcal{G}\in\mathscr{G}} \rho(A(\mathcal{L}_{\mathcal{G}}, \mathcal{X}))$, and $\Omega = \{\mathcal{X} \in \mathbb{R}^{m\times n} | \rho_\Lambda(\mathcal{X}) < 1\}$. Then it can be seen that Ω is nonempty if (A4) holds. Denote

$$\begin{aligned}
\Gamma_0(A) &= \{\eta_0 \in \mathbb{R} | \eta_0 \geq \rho(A) \ \ s.t. \ \ \|A^k\| = O(\eta_0^k)\},\\
M_0(\eta_0) &= \inf\left\{M_0 \in \mathbb{R} | M_0 \geq 1, \eta_0 \in \Gamma_0(A) \ s.t. \|A^k\| \leq M_0\eta_0^k, k \in \mathbb{N}\right\},\\
\Gamma_1(\mathcal{X}) &= \left\{\eta_1 \in \mathbb{R} | \rho_\Lambda(\mathcal{X}) \leq \eta_1 < 1 \ s.t. \| \left(A(\mathcal{L}_{\mathcal{G}}, \mathcal{X})\right)^k \| = O(\eta_1^k), \mathcal{G} \in \mathscr{G}\right\},\\
M_1(\eta_1) &= \inf\Big\{M_1 \in \mathbb{R} | M_1 \geq 1, \eta_1 \in \Gamma_1(\mathcal{X}) \ s.t.\\
&\qquad\qquad \| \left(A(\mathcal{L}_{\mathcal{G}}, \mathcal{X})\right)^k \| \leq M_1\eta_1^k, k \in \mathbb{N}, \mathcal{G} \in \mathscr{G}\Big\}.
\end{aligned}$$

Then by Lemma A.1, $\Gamma_0(A)$ is nonempty and $M_0(\eta_0)$ is well defined for any $\mathcal{X} \in \mathbb{R}^{m\times n}$. Since (A3) holds, it can be seen that each $\mathcal{G} \in \mathscr{G}$ is strongly connected. Thus, if $\mathcal{X} \in \Omega$, then $\Gamma_1(\mathcal{X})$ is nonempty and $M_1(\eta_1)$ is well defined. Denote

$$\begin{aligned}
R(\mathcal{X}, \eta_0, \eta_1) &= \max\left\{C_1C_2M_1(\eta_1), M_0(\eta_0)\eta_0^T, M_1(\eta_1)\right\},\\
\rho_1(\mathcal{X}, \eta_0, \eta_1) &= \Big(\max\{C_1C_2M_1(\eta_1), 1\}\Big)^{\frac{1}{\tau}} \eta_1^{1-\frac{T}{\tau+1}} \Big(4\sqrt{N}R(\mathcal{X}, \eta_0, \eta_1)\Big)^{\frac{T}{\tau+1}},\\
C_3(\mathcal{X}, \eta_0, \eta_1) &= \max\left\{\frac{4\sqrt{N}R(\mathcal{X}, \eta_0, \eta_1)}{(\rho_1(\mathcal{X}, \eta_0, \eta_1))^T}, C_1C_2M_1(\eta_1)\right\},\\
W(\mathcal{X}, \eta_0, \eta_1) &= T\ln\Big(4\sqrt{N}R(\mathcal{X}, \eta_0, \eta_1)\Big)\\
&\qquad + \ln\Big(\max\{C_1C_2M_1(\eta_1), 1\}\Big) + (1-T)\ln(\eta_1),\\
s(\mathcal{X}, \eta_0, \eta_1) &= \frac{W(\mathcal{X}, \eta_0, \eta_1)}{-2\ln(\eta_1)}\\
&\qquad + \frac{\sqrt{W^2(\mathcal{X}, \eta_0, \eta_1) - 4\ln(\eta_1)\ln(\max\{C_1C_2M_1(\eta_1), 1\})}}{-2\ln(\eta_1)}.
\end{aligned} \tag{3.2}$$

Let $\tau = \inf\{t_{i+1} - t_i | \mathcal{G}_{\sigma(t)}$ is strongly connected at $[t_i, t_{i+1}), i = 0, 1, \ldots\}$ as the minimum dwell time of those strongly connected digraphs of the communication topology flow $\{\mathcal{G}_{\sigma(t)}, t \in \mathbb{N}\}$. Then we have $\tau \geq 1$ and the following result.

LEMMA 3.1. *For the dynamic network* $(A, B, \mathcal{G}_{\sigma(t)})$*, suppose that* (A1)–(A5) *hold and* $\tau > \tau^*$*, where*

$$\tau^* \triangleq \inf_{\mathcal{X} \in \Omega} \inf_{\substack{\eta_0 \in \Gamma_0(A), \\ \eta_1 \in \Gamma_1(\mathcal{X})}} s(\mathcal{X}, \eta_0, \eta_1).$$

Then there exist $K \in \mathbb{R}^{m \times n}$*,* $C_3 \in \mathbb{R}$*, and* $\rho_1 \in \mathbb{R}$ *such that the distributed control protocol* (3.1) *can ensure the dynamic network to achieve synchronization exponentially fast, and* $||\delta(t)|| \leq C_3 \|\delta(0)\| \rho_1^t$*,* $t \in \mathbb{N}$*.*

The proof of Lemma 3.1 is put in the appendix.

Remark 3.2. We say that a sequence of digraphs is frequently strongly connected if it satisfies (A5). If the communication topology flow is composed of undirected graphs, then (A5) is equivalent to the frequent connectivity condition [19]. For the continuous-time linear multiagent systems with undirected communication topologies, [19] requires the dwell time τ be uniformly bounded away from zero for those connected graphs. Different from [19], Lemma 3.1 is concerned with the discrete-time case where the systems cannot be stabilized with any fast speed, so a sufficiently large dwell time for the digraphs being strongly connected is needed here.

Remark 3.3. We can derive explicit expression of τ^* for the case of first-order dynamics. Suppose that the agent dynamics are given by

$$x_i(t+1) = a x_i(t) + u_i(t), \quad i = 1, \ldots, N, \tag{3.3}$$

where $a > 1$. The communication topology flow $\{\mathcal{G}_{\sigma(t)},\ t \in \mathbb{N}\}$ is composed of undirected graphs. For this case, $A = a$, $B = 1$, and $C_1 = C_2 = 1$. By the definition of Ω, we know that if (A4) holds, then $\Omega = (\frac{a-1}{\lambda_m}, \frac{a+1}{\lambda_M})$. Since $\mathcal{G}_{\sigma(t)}$, $t \in \mathbb{N}$, are undirected, $A(\mathcal{L}_{\mathcal{G}}, \mathcal{X})$ is diagonal for all $\mathcal{G} \in \mathscr{G}$. By Lemma A.1, one can see that for any given $\eta_0 \geq a$ and $\eta_1 \geq \rho_\Lambda(\mathcal{X})$, there is $M_0(\eta_0) = M_1(\eta_1) = 1$. From (3.2) and the definition of τ^*, we get that

$$\begin{aligned}
\tau^* &= \inf_{\mathcal{X} \in \Omega} \inf_{\substack{\eta_0 \in \Gamma_0(A), \\ \eta_1 \in \Gamma_1(\mathcal{X})}} s(\mathcal{X}, \eta_0, \eta_1) \\
&= \inf_{\mathcal{X} \in \Omega} s\big(\mathcal{X}, a, \rho_\Lambda(\mathcal{X})\big) \\
&= s\left(\frac{2a}{\lambda_m + \lambda_M}, a, \rho_\Lambda\left(\frac{2a}{\lambda_m + \lambda_M}\right)\right) \\
&= \frac{T \ln(4\sqrt{N} a^T) + (1-T) \ln(a \frac{\lambda_M - \lambda_m}{\lambda_M + \lambda_m})}{-\ln(a \frac{\lambda_M - \lambda_m}{\lambda_M + \lambda_m})}.
\end{aligned} \tag{3.4}$$

Remark 3.4. Consider the dynamic network with second-order dynamics

$$\begin{cases} p_i(t+1) = p_i(t) + v_i(t), \\ v_i(t+1) = v_i(t) + u_i(t), \end{cases} \quad i = 1, \ldots, N.$$

The communication topology flow $\{\mathcal{G}_{\sigma(t)},\ t \in \mathbb{N}\}$ is composed of undirected graphs. Here, $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, and $C_1 = C_2 = 1$. Let $K = [k_1, k_2] \in \mathbb{R}^{1 \times 2}$. Take

$k_2 \in (0, \frac{4\beta}{\lambda_M})$, $k_1 = \beta k_2$, where β is a constant satisfying

$$\begin{cases} \beta \in \left(\frac{2+\sqrt{4-3\lambda_M/\lambda_m}}{4}, 1 \right), \ \lambda_M/\lambda_m < 4/3, \\ \beta \in (0,1), \ \lambda_M/\lambda_m \geq 4/3. \end{cases}$$

Then similarly to the proof of Lemma 3.1, it can be shown that if (A1)–(A5) hold and

$$\tau > \inf_{\varepsilon \in (0, W(\beta))} (1+\sqrt{2}) \cdot \frac{\left(T \ln \left(4\sqrt{N} \max\left\{ \sqrt{2N-2}(1+\frac{2}{\varepsilon})^{2N-3}, \sqrt{3}^T \right\} \right) + (2N-3) \ln \left(\sqrt{2N-2}(1+\frac{2}{\varepsilon}) \right) \right)}{-2 \ln \left(\sqrt{1-4\beta(1-\beta)\lambda_m/\lambda_M} + \varepsilon 5\sqrt{3} \right)} + \frac{T-1}{2},$$

where $W(\beta) = \frac{1-\sqrt{1-4\beta(1-\beta)\lambda_m/\lambda_M}}{5\sqrt{3}}$, then there exists $K \in \mathbb{R}^{1\times 2}$ such that the control protocol (3.1) can drive the system to synchronization exponentially fast.

Remark 3.5. For an undirected connected graph, the ratio between the minimum and the maximum nonzero Laplacian eigenvalues is a measure of its synchronizability [22]. From Remarks 3.3 and 3.4, τ^* is closely related to λ_m/λ_M, which can be viewed as the synchronizability of the communication topology flow $\{\mathcal{G}_{\sigma(t)},\ t \in \mathbb{N}\}$.

It is worth pointing out that even for the case of precise communication, our result, Lemma 3.1, is not covered by the existing literature on the synchronization of linear multiagent systems. In [20], Su and Huang proved that a linear multiagent system with a jointly connected communication topology flow can achieve synchronization by the control protocol (3.1). Compared with [20] for neutrally stable agents, the agents considered in Lemma 3.1 are unstable. Furthermore, Lemma 3.1 gives the convergence speed of synchronization errors. Below we will give an example with unstable agent dynamics and a jointly connected communication topology flow. It is found that there is no control gain to make the system achieve synchronization. From this example, we can see that for the case with unstable agent dyanmics, generally speaking, the joint connectivity condition of the communication topology flow is not enough to ensure synchronization.

Example 3.6. Consider a dynamic network with 3 agents given by

$$x_i(t+1) = ax_i(t) + u_i(t), \quad i = 1,2,3, \tag{3.5}$$

where $a > \sqrt{2+4\sqrt{3}}$. The communication topology flow $\{\mathcal{G}_{\sigma(t)},\ t \in \mathbb{N}\}$ is composed of undirected graphs. Here, $\mathcal{G}_{\sigma(t)}$ switches between two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ given by Figure 2, $\sigma(t) = 1$ for $t = 2k$, and $\sigma(t) = 2$ for $t = 2k+1$, $k \in \mathbb{N}$. It can be seen that $\{\mathcal{G}_{\sigma(t)}, t \in \mathbb{N}\}$ is jointly connected. The control protocol is given by

$$u_i(t) = h \sum_{j=1}^{3} a_{ij}(t)(x_j(t) - x_i(t)),\ h \in \mathbb{R},\ i = 1,2,3. \tag{3.6}$$

By (3.5) and the switching rule of $\sigma(t)$, we have

$$\delta(2k) = \bar{A}\delta(2(k-1)),\ k \in \mathbb{N}, \tag{3.7}$$

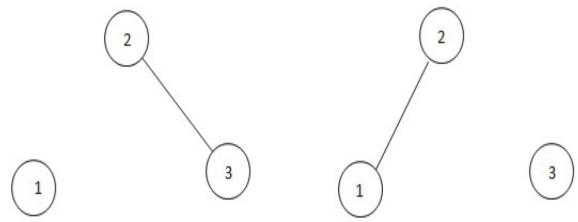

FIG. 2. $\mathcal{G}_1$ *and* $\mathcal{G}_2$.

where

$$\bar{A} = \begin{pmatrix} a(a-h) & h(a-h) & h^2 \\ ah & (a-h)^2 & h(a-h) \\ 0 & ah & a(a-h) \end{pmatrix}.$$

Let

$$\Xi = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\tilde{\delta}(t) = \Xi\delta(t)$, and $\tilde{\delta}(t) = (\tilde{\delta}_1(t), \tilde{\delta}_2(t), \tilde{\delta}_3(t))^H = (0, \delta_2(t), \delta_3(t))^H$. Then from (3.7), we know that

$$\begin{pmatrix} \tilde{\delta}_2(2k) \\ \tilde{\delta}_3(2k) \end{pmatrix} = \tilde{A}^k \begin{pmatrix} \tilde{\delta}_2(0) \\ \tilde{\delta}_3(0) \end{pmatrix},$$

where

$$\tilde{A} = \begin{pmatrix} a^2 - 3ah + h^2 & -h^2 \\ ah & a^2 - ah \end{pmatrix},$$

whose characteristic polynomial $|\lambda I - \tilde{A}| = \lambda^2 + a_1\lambda + a_0$ with $a_1 = -(2a^2 - 4ah + h^2)$ and $a_0 = a^2(a-2h)^2$. Then by the jury criteria and some direct calculation, we can prove that not all of $\tilde{A}$'s eigenvalues are inside the unit circle no matter what the control gain h takes. Noting that $\tilde{\delta}_2(0)$ and $\tilde{\delta}_3(0)$ can be given arbitrarily, one can see that $\delta(t)$ does not vanish as $t \to \infty$ for any given initial values.

Remark 3.7. In [21], Qin, Gao, and Yu studied the synchronization control of unstable agents over jointly connected network topologies when the input matrix of each agent is full of row rank (see [21, Theorem 1]). Theorem 1 of [21] requires that the maximal modulus of the eigenvalues of the agent state matrix, i.e., the most unstable mode of the agent dynamics, must be less than $1/(1 - \mu^{T_0(N-1)})^{(N-1)T_0}$, where N is the number of agents, T_0 is the upper bound of the joint-connectivity intervals, and $\mu \in (0,1)$ is a constant dependent on the weighted adjacency matrices of the topology flow. By the definition of μ, one can verify that μ must belong to $(0, \frac{\underline{\alpha}}{\overline{d}+\underline{\alpha}})$, where $\overline{d}$ is the largest diagonal element of all possible Laplacian matrices, and $\underline{\alpha}$ is the lower bound of the weighting factors. This means that though the agent dynamics of [21] can be unstable, the unstable modes should be close enough to the unit circle of the complex plane. Specifically, for the topology flow given in Example 3.6, one can verify that the unstable mode a should be less than 1.09 for meeting the conditions of Theorem 1 of [21], however, here, $a > \sqrt{2 + 4\sqrt{3}} > 2.9$. So, Example 3.6 is consistent with [21], and tells us that if the agent dynamics is sufficiently unstable, then generally the joint connectivity of the communication topology flow does not suffice for synchronization.

3.2. Communication and control protocols. Now we consider the case of finite-level quantized communication. Here, each communication channel is digital and has finite bandwidth. We have the following assumption.

(A3$'$) $\mathcal{G}_{\sigma(t)}$ is undirected, $t \in \mathbb{N}$.

Since the communication topology flow is time varying, each communication channel switches between active and inactive status discontinuously. If we use the broadcasting-type encoders and decoders proposed in [8], then the inner state of each encoder may be inconsistent with that of its neighbor's decoder due to the switching channels. Motivated by [10], we propose a dynamic encoding-decoding scheme which depends on the activeness of the communication channels: for each channel (j,i), $j \in \mathscr{N}_i$, $i = 1, \ldots, N$, if it is active, then the sender j quantizes and encodes its information and sends it to the receiver i. After receiving the information, agent i estimates j's state by the decoder. The encoder and the decoder of (j,i) update their inner states adaptively according to whether the channel (j,i) is active or not. The encoder $\Theta_{ji}(t)$ associated with (j,i) is given by

$$\Theta_{ji}(t) = \begin{cases} \xi_{ji}(0) = 0, \\ \xi_{ji}(t) = A\xi_{ji}(t-1) + a_{ij}(t)g(t)s_{ji}(t), \\ s_{ji}(t) = Q_t^{ji}\left(\dfrac{x_j(t) - A\xi_{ji}(t-1)}{g(t)}\right), \end{cases} \tag{3.8}$$

where $s_{ji}(t)$ is the output, which will be transmitted to agent i if (j,i) is active, $\xi_{ji}(t)$ is the inner state, and $g(t) > 0$ is the scaling function. Here, we take $g(t) = g_0\gamma^t$, $0 < \gamma < 1$. The decoder $\Psi_{ji}(t)$ maintained by agent i is given by

$$\Psi_{ji}(t) = \begin{cases} \hat{x}_{ji}(0) = 0, \\ \hat{x}_{ji}(t) = A\hat{x}_{ji}(t-1) + a_{ij}(t)g(t)s_{ji}(t), \end{cases} \tag{3.9}$$

where $\hat{x}_{ji}(t)$ is the output of $\Psi_{ji}(t)$, which represents the estimation of agent j's state. From (3.8) and (3.9), it can be seen that $\hat{x}_{ji}(t) = \xi_{ji}(t)$, $t \in \mathbb{N}$. Denote $E_{ji}(t) = x_j(t) - \xi_{ji}(t)$, $j \in \mathscr{N}_i$, $i = 1, \ldots, N$ as the state estimation errors. Here, the $Q_t^{ji}(\cdot)$ are uniform quantizers,

$$Q_t^{ji}(y) = \begin{cases} 0, & -\dfrac{1}{2} \le y < \dfrac{1}{2}, \\ i, & i - \dfrac{1}{2} \le y < i + \dfrac{1}{2}, \\ L_{ji}(t), & y \ge L_{ji}(t) - \dfrac{1}{2}, \\ -Q_t^{ji}(-y), & y < -\dfrac{1}{2}, \end{cases} \tag{3.10}$$

for any $y \in \mathbb{R}$, and $2L_{ji}(t) + 1$ is known as the quantization level. Denote $\Delta_{ji}(t) = \frac{x_j(t) - A\xi_{ji}(t-1)}{g(t)} - s_{ji}(t)$ as the quantization error of $Q_t^{ji}(\cdot)$. For $y = (y_1, \ldots, y_n)^H \in R^n$, let $Q_t^{ji}(y) = (Q_t^{ji}(y_1), \ldots, Q_t^{ji}(y_n))^H$, respectively.

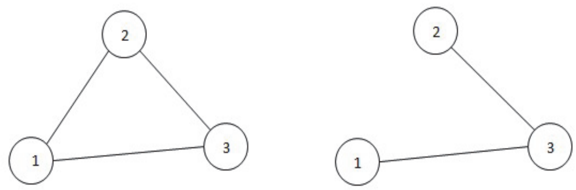

FIG. 3. $\mathcal{G}_1^{'}$ *and* $\mathcal{G}_2^{'}$.

Based on the above encoding-decoding scheme, we propose the following class of certainty-equivalence-principle-based control protocols:

$$\mathscr{U} = \left\{ u_i(t), i = 1, \dots, N, t \in \mathbb{N} \mid u_i(t) = K \sum_{j \in \mathscr{N}_i} a_{ij}(t)(\hat{x}_{ji}(t) - \xi_{ij}(t)) \right\}, \tag{3.11}$$

where $K \in \mathbb{R}^{m \times n}$ is the control gain matrix to be designed.

Remark 3.8. Here, we only design encoders and decoders for each (j, i), $j \in \mathscr{N}_i$, $i = 1, \dots, N$. According to (A2), we have $u_i(t) = K \sum_{j \in \mathscr{N}_i} a_{ij}(t)(\hat{x}_{ji}(t) - \xi_{ij}(t)) = K \sum_{j=1}^{N} a_{ij}(t)(\hat{x}_{ji}(t) - \xi_{ij}(t))$. Similarly to the certainty equivalence principle, we use the estimation of the agents' states instead of the real states to construct the control protocol in (3.11). The effectiveness of the protocols will be shown below.

3.3. Convergence analysis. Before showing the main results, we first make the following assumptions.

(A6) There exists a known constant $C_x > 0$ such that $\max_{i \in \{1, \dots, N\}} \|x_i(0)\|_\infty \leq C_x$.

(A7) For any $i \in \{1, \dots, N\}$, $j \in \mathscr{N}_i$, there exists $T_1^{ji} > 0$ such that for any $t \in \mathbb{N}$, there is $t^* \in [t, t + T_1^{ji})$ satisfying $a_{ij}(t^*) = 1$.

Remark 3.9. If (A7) holds, then we say that the channels (j, i), $j \in \mathscr{N}_i$, $i = 1, \dots, N$, are frequently active. In fact, the frequent connectivity of topologies and the frequent activeness of edges are independent. For example, the topologies in Example 3.6 are not connected at any time, but the edges (1,2) and (2,3) are frequently active with periodic $T_1^{12} = T_1^{23} = 2$. In addition, consider two graphs $\mathcal{G}_1^{'} = \{\mathscr{V}, \mathscr{E}_1^{'}\}$ and $\mathcal{G}_2^{'} = \{\mathscr{V}, \mathscr{E}_2^{'}\}$ in Figure 3, where $\mathscr{V} = \{1, 2, 3\}$, $\mathscr{E}_1^{'} = \{(1, 2), (2, 3), (1, 3)\}$, and $\mathscr{E}_2^{'} = \{(2, 3), (1, 3)\}$. If the communication topology flow switches in the following way, when $t = t_{2i}$, $\mathcal{G}_{\sigma(t)} = \mathcal{G}_1^{'}$, and when $t = t_{2i+1}$, $\mathcal{G}_{\sigma(t)} = \mathcal{G}_2^{'}$, $i = 0, 1, \dots$, where $\{t_i, i \in \mathbb{N}\}$ is a sequence of switching times and satisfies $\lim_{i \to \infty}(t_{i+1} - t_i) = \infty$, then $\mathcal{G}_{\sigma(t)}$ is always connected but the edge $(1, 2)$ is not frequently active.

The main theorem is given below.

THEOREM 3.10. *For a dynamic network* $(A, B, \mathcal{G}_{\sigma(t)})$, *assume* (A1), (A2), (A3$^{'}$), (A4)–(A6) *hold, and* $\tau > \tau^*$ *with* τ^* *defined in Lemma* 3.1. *Then there exist communication and control protocols* (3.8), (3.9), (3.10), (3.11) *such that for any given* $X(0) \in \mathscr{B}_{C_x}$, *the dynamic network achieves synchronization exponentially fast. Furthermore, if* (A7) *holds, then the exponential synchronization can be achieved with finite bits of information exchange per time step.*

Proof. It can be easily seen that Assumption (A3$^{'}$) implies (A3). Select the control gain, parameters of the encoders, decoders, and quantizers as in the following steps (a)–(d).

(a) By Lemma A.3, there exist $K^* \in \mathbb{R}^{m\times n}$ and two constants $C_3 \in \mathbb{R}$, $\rho_1 \in \mathbb{R}$ such that the solution of the linear switching system

$$w(t+1) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(t)}} \otimes BK^*)w(t)$$

has the property that $\|w(t)\| \le C_3\|w(0)\|\rho_1^t$, $t \in \mathbb{N}$. Take $K = K^*$.

(b) Take $\gamma \in (\rho_1, 1)$, where ρ_1 is defined in (a). By the proof of Lemma A.3, it can be seen that if (A1)–(A5) hold, then the value of C_3 and ρ_1 can be specified explicitly.

(c) Denote $\sup_{t\in\mathbb{N}} \max_{i=1,\dots,N} deg_i^{in}(t)$ by d^*, where $deg_i^{in}(t)$ is the in-degree of agent i at time t. Take

(3.12)
$$g_0 > \max\left\{2\|A\|_\infty C_x(\gamma-\rho_1)\Big/\Big((\gamma-\rho_1)\|A\|_\infty + 4(d^*)^2 n\sqrt{N}\|BK\|\|B\|_\infty\|K\|_\infty C_3 + 2d^*(\gamma-\rho_1)\sqrt{n}\|B\|_\infty\|K\|_\infty\Big), \frac{2C_x(\gamma-\rho_1)}{\sqrt{n}d^*\|BK\|}\right\}.$$

(d) Denote

$$\tilde{M} = \frac{\|A\|_\infty}{2\gamma} + \tilde{H},$$

where

$$\tilde{H} = \frac{2(d^*)^2 n\sqrt{N}\|BK\|\|B\|_\infty\|K\|_\infty C_3}{\gamma(\gamma-\rho_1)} + \frac{d^*\sqrt{n}\|B\|_\infty\|K\|_\infty}{\gamma}.$$

For $i \in \{1,\dots,N\}, j \in \mathcal{N}_i$, take $L_{ji}(1) = \lceil \tilde{M} - \frac{1}{2}\rceil$, and for $t \ge 2$,

$$L_{ji}(t) = \begin{cases} \left\lceil \frac{\|A\|_\infty}{\gamma}\left(L_{ji}(t-1)+\frac{1}{2}\right) + \tilde{H} - \frac{1}{2}\right\rceil, & a_{ij}(t-1) = 0 \\ \lceil \tilde{M} - \frac{1}{2}\rceil, & a_{ij}(t-1) = 1 \end{cases}. \quad (3.13)$$

By (2.1), we have

$$X(l+1) = (I_N \otimes A)X(l) + (I_N \otimes B)U(l),\ l \in \mathbb{N}. \quad (3.14)$$

For any $l \in \mathbb{N}$, denote $a_i(l) = (a_{i1}(l),\dots,a_{iN}(l))^H$, $\alpha_i(l) = (a_{1i}(l),\dots,a_{Ni}(l))^H$, $\Sigma_1(l) = diag(a_1^H(l),\dots,a_N^H(l))$, $\Sigma_2(l) = diag(\alpha_1^H(l),\dots,\alpha_N^H(l))$, and

$$\hat{E}(l) = \left(E_{11}^H(l), E_{12}^H(l),\dots,E_{1N}^H(l), E_{21}^H(l),\dots,E_{NN}^H(l)\right)^H,$$
$$\bar{E}(l) = \left(E_{11}^H(l), E_{21}^H(l),\dots,E_{N1}^H(l), E_{12}^H(l),\dots,E_{NN}^H(l)\right)^H.$$

Then by (3.11), we have $U(l) = (-\mathcal{L}_{\mathcal{G}_{\sigma(l)}} \otimes K)X(l) + (\Sigma_1(l)\otimes K)\hat{E}(l) - (\Sigma_2(l)\otimes K)\bar{E}(l)$. This together with (3.14) leads to

$$\begin{aligned} X(l+1) = {} & (I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(l)}} \otimes BK)X(l) \\ & + (\Sigma_1(l)\otimes BK)\hat{E}(l) - (\Sigma_2(l)\otimes BK)\bar{E}(l),\ l \in \mathbb{N}. \end{aligned} \quad (3.15)$$

By Assumption (A3$'$), we have $a_{ij}(l) = a_{ji}(l)$. Then it can be seen that for any $l \in \mathbb{N}$,

$$(\mathbf{1}_N^H \otimes I_n)\left[(\Sigma_1(l)\otimes BK)\hat{E}(l) - (\Sigma_2(l)\otimes BK)\bar{E}(l)\right] = 0.$$

This together with $\mathbf{1}^H \mathcal{L}_{\mathcal{G}_{\sigma(l)}} = 0$ leads to

$$(\mathbf{1}^H \otimes I_n)X(l+1) = A(\mathbf{1}^H \otimes I_n)X(l),\ l \in \mathbb{N}. \tag{3.16}$$

From (3.15) and (3.16), we have

$$\begin{aligned}\delta(l) &= X(l) - \frac{1}{N}(\mathbf{1}\mathbf{1}^T \otimes I_n)X(l)\\ &= (I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(l-1)}} \otimes BK)\delta(l-1)\\ &\quad + (\Sigma_1(l-1) \otimes BK)\hat{E}(l-1) - (\Sigma_2(l-1) \otimes BK)\bar{E}(l-1)\\ &= \prod_{h=0}^{l-1}(I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(h)}} \otimes BK)\delta(0)\\ &\quad + \sum_{h=0}^{l-1}\left[\prod_{j=h+1}^{l-1}(I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(j)}} \otimes BK)\Big((\Sigma_1(h) \otimes BK)\hat{E}(h)\right.\\ &\qquad\qquad \left.- (\Sigma_2(h) \otimes BK)\bar{E}(h)\Big)\right],\ l \in \mathbb{N}.\end{aligned} \tag{3.17}$$

Let $z(l) = \prod_{h=0}^{l-1}(I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(h)}} \otimes BK)\delta(0)$, $l > 0$ and $z(0) = \delta(0)$. Then, $z(l) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(l-1)}} \otimes BK)z(l-1)$, and $(\mathbf{1}_N^H \otimes I_n)z(l) = 0$, $l \in \mathbb{N}$. By Lemma A.3, one can see that

$$\|z(l)\| \le C_3\|z(0)\|\rho_1^l = C_3\|\delta(0)\|\rho_1^l,\ l \in \mathbb{N}. \tag{3.18}$$

For a positive integer h, let $v(l) = \prod_{j=h+1}^{l-1}(I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(j)}} \otimes BK)\big[(\Sigma_1(h) \otimes BK)\hat{E}(h) - (\Sigma_2(h) \otimes BK)\bar{E}(h)\big]$, $l > h+1$, and $v(h+1) = (\Sigma_1(h) \otimes BK)\hat{E}(h) - (\Sigma_2(h) \otimes BK)\bar{E}(h)$, and $v(l) = 0$ when $0 \le l < h+1$. Then by the definition of $v(l)$, we have

$$v(l) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(l-1)}} \otimes BK)v(l-1). \tag{3.19}$$

Since $(\mathbf{1}_N^H \otimes I_n)[(\Sigma_1(l) \otimes BK)\hat{E}(l) - (\Sigma_2(l) \otimes BK)\bar{E}(l)] = 0$, it can be seen that $(\mathbf{1}_N^H \otimes I_n)v(l) = 0$, $l \in \mathbb{N}$. By Lemma A.3, we know that

$$\|v(l)\| \le 2\sqrt{nN}d^* C_3 \rho_1^{l-1-h}\|BK\| \max_{i,j|a_{ij}(h)=1}\|E_{ij}(h)\|. \tag{3.20}$$

From (3.17), (3.18), and (3.20), we have

$$\|\delta(l)\| \le C_3\rho_1^l\|\delta(0)\| + 2\sqrt{nN}d^*\|BK\|C_3\sum_{h=0}^{l-1}\rho_1^{l-1-h}\max_{i,j|a_{ij}(h)=1}\|E_{ij}(h)\|,\ l \in \mathbb{N}. \tag{3.21}$$

Next we prove that by selecting the parameters of the communication and control protocols as in (a)–(d), the quantizers will never be saturated. For any (i,j), $i \in \mathscr{N}_j$, $j \in \{1,\dots,N\}$, at the time $l = 1$ we have

$$\frac{x_i(1) - A\xi_{ij}(0)}{g(1)} = \frac{x_i(1)}{g_0\gamma}. \tag{3.22}$$

By (2.1) and (3.11), we have $x_i(1) = Ax_i(0)$. Then from (3.22), we know that

$$\left\|\frac{x_i(1) - A\xi_{ij}(0)}{g(1)}\right\|_\infty \le \frac{\|A\|_\infty C_x}{g_0\gamma}.$$

Noting (3.12), we have $\left\|(x_i(1) - A\xi_{ij}(0))/g(1)\right\|_\infty \le \tilde{M} \le L_{ij}(1) + \frac{1}{2}$. So, the quantizers are not saturated at $l = 1$.

Assuming that the quantizers are not saturated at $l = 1, 2, \ldots, t-1$, it can be seen that $\|\Delta_{ij}(l)\|_\infty \le \frac{1}{2}$, $l = 1, \ldots, t-1$. Now, consider the time $l = t$. By (2.1), we have

$$\begin{aligned} x_i(t) - A\xi_{ij}(t-1) &= Ax_i(t-1) + Bu_i(t-1) - A\xi_{ij}(t-1) \\ &= AE_{ij}(t-1) + Bu_i(t-1). \end{aligned} \tag{3.23}$$

By (3.11), we know that

$$u_i(l) = K\sum_{i=1}^N a_{ij}(l)(\delta_j(l) - \delta_i(l)) + K\sum_{j=1}^N a_{ij}(l)E_{ij}(l) - K\sum_{j=1}^N a_{ji}(l)E_{ji}(l),\ l \in \mathbb{N},$$

which leads to

$$\|u_i(l)\|_\infty \le 2d^*\|K\|_\infty\|\delta(l)\| + 2d^*\|K\|_\infty \max_{i,j|a_{ij}(l)=1}\|E_{ij}(l)\|,\ l \in \mathbb{N}. \tag{3.24}$$

We first give an upper bound of $\max_{i,j|a_{ij}(l)=1}\|E_{ij}(l)\|,\ l = 1, \ldots, t-1$. From (3.8), we have

$$E_{ij}(l) = (1 - a_{ij}(l))(x_i(l) - A\xi_{ij}(l-1)) + a_{ij}(l)g(l)\Delta_{ij}(l),\ l \in \mathbb{N}. \tag{3.25}$$

By (3.25), we can see that $E_{ij}(l) = g(l)\Delta_{ij}(l)$ if $a_{ij}(l) = 1$, and $E_{ij}(l) = x_i(l) - A\xi_{ij}(l-1)$ otherwise. Since $\|\Delta_{ij}(l)\|_\infty \le \frac{1}{2}$, $l \le t-1$, and noting (3.25), we have $\max_{i,j|a_{ij}(l)=1}\|E_{ij}(l)\| \le \frac{\sqrt{n}g_0}{2}\gamma^l$, $l = 1, \ldots, t-1$. This together with (3.12) and (3.21) leads to

$$\|\delta(l)\| \le \frac{n\sqrt{N}d^*\|BK\|C_3g_0}{\gamma - \rho_1}\gamma^l, \quad l = 1, \ldots, t-1. \tag{3.26}$$

By (3.26), (3.23), and (3.24), it can be seen that

$$\begin{aligned} \|x_i(t) - A\xi_{ij}(t-1)\|_\infty \le \|A\|_\infty\|E_{ij}(t-1)\|_\infty& \\ + \frac{2(d^*)^2g_0n\sqrt{N}C_3\|BK\|\|B\|_\infty\|K\|_\infty}{\gamma - \rho_1}\gamma^{t-1}& \\ + d^*\sqrt{n}g_0\|B\|_\infty\|K\|_\infty\gamma^{t-1}& \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \left\|\frac{x_i(t) - A\xi_{ij}(t-1)}{g(t)}\right\|_\infty &\le \|A\|_\infty\frac{\|E_{ij}(t-1)\|_\infty}{g(t)} + \frac{d^*\sqrt{n}\|B\|_\infty\|K\|_\infty}{\gamma} \\ &\quad + \frac{2(d^*)^2n\sqrt{N}C_3\|BK\|\|B\|_\infty\|K\|_\infty}{\gamma(\gamma - \rho_1)}. \end{aligned} \tag{3.28}$$

If $a_{ij}(t-1) = 1$, then $E_{ij}(t-1) = g(t-1)\Delta_{ij}(t-1)$ and $\|E_{ij}(t-1)\|_\infty \le \frac{g_0}{2}\gamma^{t-1}$. From (3.28), we have

$$\begin{aligned} \left\|\frac{x_i(t) - A\xi_{ij}(t-1)}{g(t)}\right\|_\infty &\le \frac{\|A\|_\infty}{2\gamma} + \frac{2(d^*)^2n\sqrt{N}C_3\|BK\|\|B\|_\infty\|K\|_\infty}{\gamma(\gamma - \rho_1)} \\ &\quad + \frac{d^*\sqrt{n}\|B\|_\infty\|K\|_\infty}{\gamma} \\ &= \tilde{M} \le L_{ij}(t) + \frac{1}{2}. \end{aligned} \tag{3.29}$$

Thus, we can see that if $a_{ij}(t-1)=1$, the quantizer $Q_l^{ij}(\cdot)$ is not saturated at time $l=t$.

If $a_{ij}(t-1)=0$, then, by (3.25), $E_{ij}(t-1)=x_i(t-1)-A\xi_{ij}(t-2)$. Since the quantizers are not saturated before t, we can see that $\|x_i(t-1)-A\xi_{ij}(t-2)\|_\infty \le g(t-1)(L_{ij}(t-1)+\frac{1}{2})$. This together with (3.28) leads to

$$\begin{aligned}\left\|\frac{x_i(t)-A\xi_{ij}(t-1)}{g(t)}\right\|_\infty &\le \frac{\|A\|_\infty}{\gamma}\left(L_{ij}(t-1)+\frac{1}{2}\right)+\frac{d^*\sqrt{n}\|B\|_\infty\|K\|_\infty}{\gamma}\\ &\quad+\frac{2(d^*)^2n\sqrt{N}C_3\|BK\|\|B\|_\infty\|K\|_\infty}{\gamma(\gamma-\rho_1)}\\ &=\frac{\|A\|_\infty}{\gamma}\left(L_{ij}(t-1)+\frac{1}{2}\right)+\tilde{H}\\ &\le L_{ij}(t)+\frac{1}{2}.\end{aligned} \tag{3.30}$$

From (3.30), we can see that if $a_{ij}(t-1)=0$, then the quantizer $Q_l^{ij}(\cdot)$ is also not saturated at $l=t$. By induction, we know that the quantizers will never be saturated, so $\|\Delta_{ij}(t)\|_\infty \le \frac{1}{2}$, $t\in\mathbb{N}$, which together with (3.21) and (3.26) leads to

$$\|\delta(t)\| \le \frac{n\sqrt{N}d^*\|BK\|C_3g_0}{\gamma-\rho_1}\gamma^t = O(\gamma^t),\ t\in\mathbb{N}. \tag{3.31}$$

From the above, we can see that multiagent systems achieve synchronization exponentially fast.

Furthermore, if (A7) holds, let $T_1=\max_{i,j}T_1^{ji}$. Then we have

$$\begin{aligned}&\sup_{t\in\mathbb{N}}\max_{j\in\mathcal{N}_i,i=1,\dots,N}L_{ji}(t)\\ &\le\left(\frac{\|A\|_\infty}{\gamma}\right)^{T_1}\left(\tilde{M}+\frac{1}{2}\right)+\frac{(\|A\|_\infty/\gamma)^{T_1}-1}{\|A\|_\infty/\gamma-1}\left(\frac{\|A\|_\infty}{2\gamma}+\tilde{H}+\frac{1}{2}\right)<\infty.\end{aligned} \tag{3.32}$$

So, exponential synchronization can be achieved with finite bits of information exchange. ∎

Remark 3.11. Lemma 3.1 and Theorem 3.10 are consistent with the case of fixed topologies in [9]. In fact, if the topology flow is a fixed digraph and strongly connected, then $\tau=+\infty$, which definitely satisfies $\tau>\tau^*$. From the numerical examples in section 4, we can see that the estimation for the minimum dwell time required is still conservative and might be improved in future investigation.

By Theorem 3.10 and Remark 3.3, we have the following corollary.

COROLLARY 3.12. *For a dynamic network* $(a,1,\mathcal{G}_{\sigma(t)}), a>1$, *assume that* (A1), (A2), (A3$'$), (A4)–(A7) *hold, and* $\tau>\tau^*$, *where* τ^* *is given in Remark* 3.3. *Then there exist communication and control protocols* (3.8), (3.9), (3.10), (3.11) *such that for any given* $X(0)\in\mathscr{B}_{C_x}$, *the dynamic networks achieves synchronization exponentially with less than*

$$\begin{aligned}\Bigg[&a^{T_1}\left[\frac{a+1}{2}+\frac{8a^2(d^*)^2\sqrt{N}C_3}{(1-\rho_1)(\lambda_m+\lambda_M)^2}+\frac{2ad^*}{\lambda_m+\lambda_M}\right]\\ &+\frac{a^{T_1}-1}{a-1}\left[\frac{a+1}{2}+\frac{8a^2(d^*)^2\sqrt{N}C_3}{(1-\rho_1)(\lambda_m+\lambda_M)^2}+\frac{2ad^*}{\lambda_m+\lambda_M}\right]+\frac{1}{2}\Bigg]\end{aligned}$$

bits of information exchange per step, where $C_3 = \frac{4\sqrt{N}a^T}{\rho_1^T}$, $\rho_1 = (a\frac{\lambda_M-\lambda_m}{\lambda_M+\lambda_m})^{1-\frac{T}{\tau+1}} \cdot (4\sqrt{N}a^T)^{\frac{T}{\tau+1}}$*, and* $\lceil x \rceil$ *deotes the minimum integer equal to or larger than* x *for any given real number* x*.*

Proof. By Remark 3.3 and Lemma A.2, we know that there exist $X = \frac{2a}{\lambda_M+\lambda_m}$, $\eta_0 = a$, $\eta_1 = \eta_2 = a\frac{\lambda_M-\lambda_m}{\lambda_M+\lambda_m}$ such that $\rho_1(X, \eta_0, \eta_1, \eta_2) = \rho_1 < 1$. By the proof of Lemma A.3 and noting (A.29), we know that for the class of switching systems

$$w(t+1) = (I_N \otimes a - \mathcal{L}_{\mathcal{G}_{\sigma(t)}} \otimes K^*)w(t),$$

where $w(t) \in \mathbb{R}^N, t \in \mathbb{N}$, there is $\|w(t)\| \leq C_3\|w(0)\|\rho_1^t$, $t \in \mathbb{N}$, where

$$K^* = \frac{2a}{\lambda_M + \lambda_m}. \tag{3.33}$$

Take $K = K^*$. Similarly to the proof of Theorem 3.10 and the method of parameter selection, and noting (3.33), we have

$$\sup_{t\in\mathbb{N}} \max_{j\in\mathscr{N}_i, i=1,\ldots,N} L_{ji}(t) \leq f(\gamma),$$

where

$$\begin{aligned} f(\gamma) = \left(\frac{a}{\gamma}\right)^{T_1} &\left[\frac{a}{2\gamma} + \frac{1}{2} + \frac{8a^2(d^*)^2\sqrt{N}C_3}{\gamma(\gamma-\rho_1)(\lambda_m+\lambda_M)^2} + \frac{2ad^*}{\gamma(\lambda_m+\lambda_M)}\right] \\ &+ \frac{(a/\gamma)^{T_1-1}}{a/\gamma - 1}\left[\frac{8a^2(d^*)^2\sqrt{N}C_3}{\gamma(\gamma-\rho_1)(\lambda_m+\lambda_M)^2} + \frac{2ad^*}{\gamma(\lambda_m+\lambda_M)} + \frac{1}{2} + \frac{a}{2\gamma}\right]. \end{aligned}$$

Noting that $f(\gamma)$ is a continuous function of γ, we know that there exist a γ^* such that $\sup_{t\in\mathbb{N}} \max_{j\in\mathscr{N}_i,i=1,\ldots,N} L_{ji}(t) \leq f(\gamma^*) < \lim_{\gamma\to 1} f(\gamma) + \frac{1}{2} = a^{T_1}[\frac{a+1}{2} + \frac{8a^2(d^*)^2\sqrt{N}C_3}{(1-\rho_1)(\lambda_m+\lambda_M)^2} + \frac{2ad^*}{\lambda_m+\lambda_M}] + \frac{a^{T_1}-1}{a-1}[\frac{8a^2(d^*)^2\sqrt{N}C_3}{(1-\rho_1)(\lambda_m+\lambda_M)^2} + \frac{2ad^*}{\lambda_m+\lambda_M} + \frac{a+1}{2}] + \frac{1}{2}$. Take $\gamma = \gamma^*$ and g_0 as in (3.12). Then by Theorem 3.10, we get the conclusion. ∎

4. Numerical example. In this section, we give some numerical examples to illustrate the effectiveness of our protocols. We may see that for the dynamic networks with unstable agent dynamics, a large enough dwell time for connected topologies and large enough number of quantization levels are needed for achieving synchronization. It can also be seen that the results derived in this paper may be conservative in practise. How to reduce the conservativeness of the results will be a future interesting topic.

Consider a multiagent system with 3 agents, each with the following dynamics:

$$x_i(t+1) = 1.2x_i(t) + u_i(t)\ , i = 1, 2, 3.$$

The initial states $x_1(0), x_2(0)$, and $x_3(0)$ are randomly selected in $(-6, 6)$. There are two undirected graphs $\mathcal{G}_1^{''} = \{\mathscr{V}, \mathscr{E}_1^{''}\}$ and $\mathcal{G}_2^{''} = \{\mathscr{V}, \mathscr{E}_2^{''}\}$, where $\mathscr{V} = \{1, 2, 3\}$, $\mathscr{E}_1^{''} = \{(1, 2)\}$, and $\mathscr{E}_2^{''} = \{(1, 2), (2, 3)\}$. See Figure 4. The communication topology flow $\{\mathcal{G}_{\sigma(t)}, t \in \mathbb{N}\}$ switches between $\mathcal{G}_1^{''}$ and $\mathcal{G}_2^{''}$ in the following way: At time $t = 0$, $\sigma(0) = 1$ and $\sigma(t)$ keeps in mode 1 for two steps, then it switches to mode 2 at $t = 2$ and keeps for τ steps, then it switches back to mode 1 and so on. It can be seen that (A1)–(A7) hold with $T = 2$, $C_x = 6$, $T_1^{12} = 1$, and $T_1^{23} = 3$.

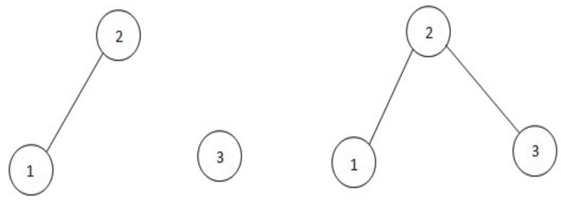

FIG. 4. $\mathcal{G}_1''$ *and* $\mathcal{G}_2''$.

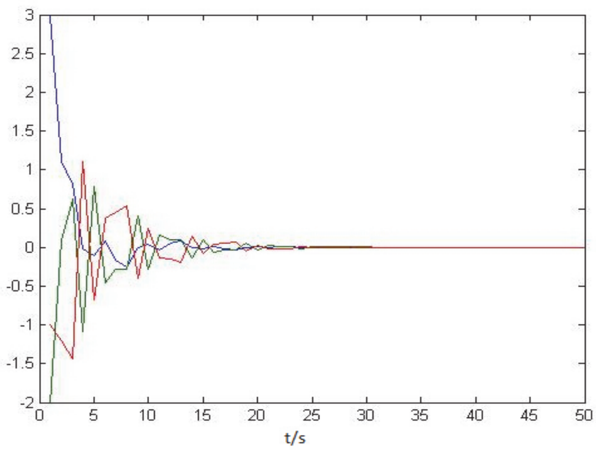

FIG. 5. *The trajectories of the synchronization errors when* $\tau = 3$.

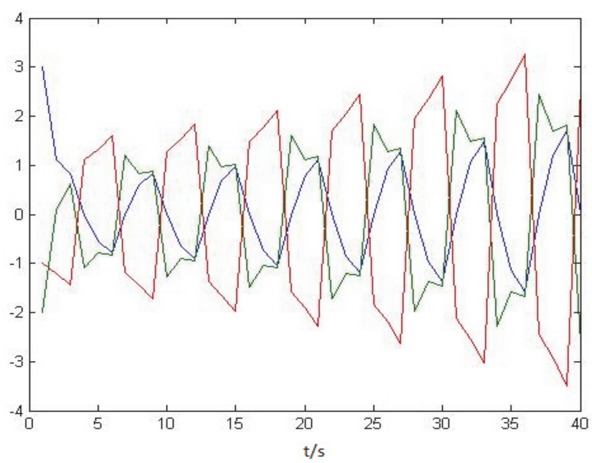

FIG. 6. *The trajectories of the synchronization errors when* $\tau = 1$.

For the case of precise communication, the control protocol takes the form of (3.1). Take the control gain $K = 0.5$; by Lemma 3.1 it can be derived that $\tau^* = 10$ for this case. Here, we first take $\tau = 3$, and the trajectories of synchronization errors are given in Figure 5. It can be seen that the τ^* derived from Lemma 3.1 is conservative. Taking $\tau = 1$, the trajectories of synchronization errors are given in Figure 6. It is shown that the synchronization may fail if the dwell time is too small.

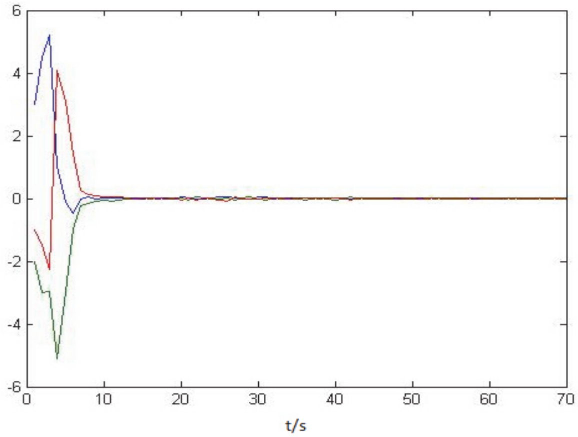

FIG. 7. *The trajectories of the synchronization errors when* $M^* = 20, H^* = 10$.

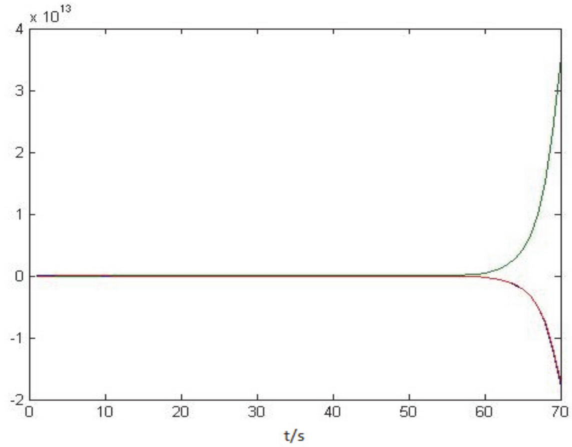

FIG. 8. *The trajectories of the synchronization errors when* $M^* = 8$, $H^* = 4$.

For the case of finite-level quantized communication, the control protocol takes the form of (3.11). Take $K = 0.5$, $g_0 = 0.16$, and $\gamma = 0.98$. The number of quantization levels is adjusted in the following way. Take $L_{ji}(1) = \lceil M^* - \frac{1}{2} \rceil$, and for $t \geq 2$,

$$L_{ji}(t) = \begin{cases} \left\lceil \frac{1.2}{0.98} \left(L_{ji}(t-1) + \frac{1}{2} \right) + H^* - \frac{1}{2} \right\rceil, & a_{ij}(t-1) = 0, \\ \lceil M^* - \frac{1}{2} \rceil, & a_{ij}(t-1) = 1. \end{cases} \tag{4.1}$$

First, let $M^* = 20$ and $H^* = 10$ in (4.1). The trajectories of synchronization errors are given in Figure 7. Second, let $M^* = 8$ and $H^* = 4$ in (3.13). The trajectories of synchronization errors are given in Figure 8. It can be seen that if the number of quantization levels is too small, the protocols may also fail for synchronization.

5. Conclusion. In this paper, we studied the synchronization of discrete-time linear multiagent systems with finite communication data rate and switching topology flows. We proposed a class of quantized observer-based encoding-decoding communication schemes and a class of certainty-equivalence-principle-based control protocols.

To avoid the information mismatching due to switching topologies, adaptive encoders and decoders are designed with the number of quantization levels dynamically adjusted according to the status of the associated channels for the last time step. It is shown that if the pairs of the agents' state matrices and the control matrices multiplied by Laplaian eigenvalues of the weakly connected components are simultaneously stabilizable, the communication topology flow is frequently connected, and the dwell time is sufficiently large for the digraphs being strongly connected, then there exist such protocols leading to synchronization exponentially fast. Furthermore, if the channels are frequently active, then finite bits of information exchange per step can guarantee the exponential synchronization. It is shown that the data rate is closely related to the unstable mode of agent dynamics, the number of agents, the frequency of graph connectivity, and the ratio of the smallest and largest nonzero Laplacian eigenvalues of the connected components.

In this paper, only switching undirected graphs are considered for the case with quantized communication. It is more interesting to consider the case with general switching digraphs. First, the results could be extended to the case with switching balanced digraphs by using the methodology of [11]. However, the Hadamard product and associated properties used in [11] would not be applicable and one would need to develop effective tools to simplify and analyze the closed-loop dynamics. It is more challenging to consider the case with general switching digraphs, since there would be no information loop in the graph, the error-compensation-type control protocols in our paper and [10], [11] will not work any longer. These remain as interesting open problems for future investigation. Also, the cases with partial measurable states, time delay, and random switching topology are interesting topics.

Appendix A. The following semimartingale convergence theorem can be found in [23].

LEMMA A.1 (see [23]). *For any square matrix $A \in \mathbb{C}^{n\times n}$, there exist $M \geq 1$, $\eta \geq \rho(A)$ such that*

$$\|A^k\| \leq M\eta^k,\ \ k \in \mathbb{N},$$

where M depends on η and the dimension n but is independent of $\rho(A)$. Especially, if A is a symmetric matrix, then M and η can be equal to 1 and $\rho(A)$, respectively.

LEMMA A.2. *Assume that* (A1)–(A5) *hold and $\tau > \tau^*$ with τ^* defined in Lemma* 3.1. *Then the set $\Upsilon = \{(\mathcal{X}, \eta_0, \eta_1) | \mathcal{X} \in \Omega, \eta_0 \in \Gamma_0(A), \eta_1 \in \Gamma_1(\mathcal{X}), \rho_1(\mathcal{X}, \eta_0, \eta_1) < 1, C_1C_2M_1(\eta_1)\eta_1^\tau < 1\}$ is nonempty.*

Proof. By $\tau > \tau^*$, we know that there exist $\mathcal{X} \in \Omega, \eta_0 \in \Gamma_0(A)$, and $\eta_1 \in \Gamma_1(\mathcal{X})$ such that $\tau > s(\mathcal{X}, \eta_0, \eta_1)$, which is equivalent to

$$\text{(A.1)} \qquad \Big(\max\{C_1C_2M_1(\eta_1), 1\}\Big)^{\frac{1}{\tau}} \eta_1^{1-\frac{T}{\tau+1}} \cdot \Big(4\sqrt{N}R(\mathcal{X}, \eta_0, \eta_1)\Big)^{\frac{T}{\tau+1}} < 1.$$

By the definition of $\rho_1(\mathcal{X}, \eta_0, \eta_1)$ and noting (A.1), it can be seen that there exist $\mathcal{X} \in \Omega, \eta_0 \in \Gamma_0(A)$, and $\eta_1 \in \Gamma_1(\mathcal{X})$ such that $\rho_1(\mathcal{X}, \eta_0, \eta_1) < 1$. By the definition of $\Gamma_1(\mathcal{X})$, we know that for any given $\eta_1 \in \Gamma_1(\mathcal{X})$, there is $\eta_1 < 1$. By the definition of $R(\mathcal{X}, \eta_0, \eta_1)$, we know that $R(\mathcal{X}, \eta_0, \eta_1) \geq 1$. This together with the definition of $\rho_1(\mathcal{X}, \eta_0, \eta_1)$ leads to $\rho_1(\mathcal{X}, \eta_0, \eta_1) > (C_1C_2M_1(\eta_1))^{\frac{1}{\tau}}\eta_1$. Noting that $\rho_1(\mathcal{X}, \eta_0, \eta_1) < 1$, we have $C_1C_2M_1(\eta_1)\eta_1^\tau < 1$. From the above, we can see that Υ is nonempty. ∎

Proof of Lemma 3.1. Consider an interval $[t_i, t_{i+1})$ during which the communication topology keeps fixed. By (3.1), we know that $U(t) = -(\mathcal{L}_{\mathcal{G}_{\sigma(t)}} \otimes K)X(t)$. From (2.1), we have

$$X(t+1) = (I \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(t)}} \otimes BK)X(t). \tag{A.2}$$

From (A.2) and noting that $\mathcal{G}_{\sigma(t)}$ is balanced, we have

$$\bar{x}(t+1) = \left(\frac{1}{N}\mathbf{1}_N^H \otimes A\right) X(t) = A\bar{x}(t). \tag{A.3}$$

From (A.2) and (A.3), we know that

$$\delta(t+1) = (I \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(t)}} \otimes BK)\delta(t). \tag{A.4}$$

Denote $\Upsilon = \{(\mathcal{X}, \eta_0, \eta_1) | \mathcal{X} \in \Omega, \eta_0 \in \Gamma_0(A), \eta_1 \in \Gamma_1(\mathcal{X}), \rho_1(\mathcal{X}, \eta_0, \eta_1,) < 1, C_1C_2 \cdot M_1(\eta_1)\eta_1^{\tau} < 1\}$. By Lemma A.2, we know that the set Υ is nonempty. Take $(K, \eta_0, \eta_1) \in \Upsilon$.

Now consider two cases.

(1) $\mathcal{G}_{\sigma(t)}$ is strongly connected on $[t_i, t_{i+1})$.

Transform $\mathcal{L}_{\mathcal{G}_{\sigma(t)}}$ to its Jordan canonical with the transformation matrix $\Phi_{\mathcal{G}_{\sigma(t)}}$. Denote $\tilde{\delta}(t) = (\Phi_{\mathcal{G}_{\sigma(t)}} \otimes I_n)\delta(t)$. By $(A.4)$, we have

$$\tilde{\delta}(t+1) = \left(I_N \otimes A - diag\left(0, J_2^{\mathcal{G}_{\sigma(t)}}, \ldots, J_{k_{\mathcal{G}_{\sigma(t)}}}^{\mathcal{G}_{\sigma(t)}}\right) \otimes BK\right)\tilde{\delta}(t). \tag{A.5}$$

Denote the first n elements of $\tilde{\delta}(t)$ by $\tilde{\delta}_1(t)$, and the others by $\tilde{\delta}_2(t)$. Since the first row of $\Phi_{\mathcal{G}_{\sigma(t)}}$ is $\mathbf{1}_N^H$, $t \in \mathbb{N}$, we know that $\tilde{\delta}_1(t) = \sum_{i=1}^N \delta_i(t) \equiv 0$. By the definition of $A(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}, K)$, it can be seen that $A(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}, K) = I_{N-1} \otimes A - diag(J_2^{\mathcal{G}_{\sigma(t)}}, \ldots, J_{k_{\mathcal{G}_{\sigma(t)}}}^{\mathcal{G}_{\sigma(t)}}) \otimes BK$ and $\rho(A(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}, K)) = \max_{1 \le i \le k_{\mathcal{G}_{\sigma(t)}}} \rho(A - \lambda_i(\mathcal{L}_{\mathcal{G}_{\sigma(t)}})BK)$. Since $\mathcal{G}_{\sigma(t)}$ is strongly connected on $[t_i, t_{i+1})$, we have $\lambda_i(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}) \in \Lambda$, $t \in [t_i, t_{i+1})$. So $\rho(A(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}, K)) \le \rho_\Lambda(K) < 1$, $t \in [t_i, t_{i+1})$.

By (A.5), we know that

$$\tilde{\delta}_2(t+1) = A(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}, K)\tilde{\delta}_2(t) \tag{A.6}$$

and

$$\tilde{\delta}_2(t_{i+1}) = (A(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}, K))^{t_{i+1}-t_i}\tilde{\delta}_2(t_i). \tag{A.7}$$

By the definition of $\Gamma_1(K)$, we know that

$$\|A(\mathcal{L}_{\mathcal{G}_{\sigma(t)}}, K)^{t_{i+1}-t_i}\| \le M_1(\eta_1)\eta_1^{t_{i+1}-t_i}. \tag{A.8}$$

This together with (A.7) leads to

$$\|\tilde{\delta}_2(t_{i+1})\| \le M_1(\eta_1)\eta_1^{t_{i+1}-t_i}\|\tilde{\delta}_2(t_i)\|. \tag{A.9}$$

Then from (A.9) and noting the definitions of C_1 and C_2, we can see that

$$\|\delta(t_{i+1})\| \le C_2\|\tilde{\delta}_2(t_{i+1})\| \le C_1C_2M_1(\eta_1)\eta_1^{t_{i+1}-t_i}\|\delta(t_i)\|. \tag{A.10}$$

(2) $\mathcal{G}_{\sigma(t)}$ is not strongly connected on $[t_i, t_{i+1})$.

Divide the graph $\mathcal{G}_{\sigma(t)}$ into its weakly connected components denoted by $\mathcal{G}_{\mu,\sigma(t)}, \mu = 1, \ldots, s_{\sigma(t)}$. It can be seen that $\mathcal{G}_{\mu,\sigma(t)} \in \mathscr{G}$ and all the $\mathcal{G}_{\mu,\sigma(t)}$ are balanced digraphs. Denote the vertex set of $\mathcal{G}_{\mu,\sigma(t)}$ by $V_{\mu,\sigma(t)}$, and its module $|V_{\mu,\sigma(t)}|$ by $n_{\mu,\sigma(t)}$. Relabel the agents according to the components they belong to: for any $\mu \in \{1, \ldots, s_{\sigma(t)}\}$, denote the states of the agents in $\mathcal{G}_{\mu,\sigma(t)}$ by $x_{\mu_1}(t), \ldots, x_{\mu_{n_{\mu,\sigma(t)}}}(t)$. Denote $X_\mu(t) = (x^H_{\mu_1}(t), \ldots, x^H_{\mu_{n_{\mu,\sigma(t)}}}(t))^H, U_\mu(t) = (u^H_{\mu_1}(t), \ldots, u^H_{\mu_{n_{\mu,\sigma(t)}}}(t))^H$, and $\bar{x}_\mu(t) = (\frac{1}{n_{\mu,\sigma(t)}}\mathbf{1}^H_{n_{\mu,\sigma(t)}} \otimes I_n)X_\mu(t)$.

By (3.1), for any given $\mathcal{G}_{\mu,\sigma(t)}, \mu = 1, \ldots, s_{\sigma(t)}$, we have

$$U_\mu(t) = -(\mathcal{L}_{\mathcal{G}_{\mu,\sigma(t)}} \otimes K)X_\mu(t)$$

and

$$\text{(A.11)} \qquad \begin{aligned}(\mathbf{1}^H_{n_{\mu,\sigma(t)}} \otimes I_m)U_\mu(t) &= -(\mathbf{1}^H_{n_{\mu,\sigma(t)}} \otimes I_m)(\mathcal{L}_{\mathcal{G}_{\mu,\sigma(t)}} \otimes K)X_\mu(t)\\ &= 0.\end{aligned}$$

By (2.1), we know that

$$\text{(A.12)} \qquad x_{\mu_i}(t+1) = Ax_{\mu_i}(t) + Bu_{\mu_i}(t),\ i = 1, \ldots, \mu_{n_{\mu,\sigma(t)}}.$$

This together with (A.11) leads to:

$$\text{(A.13)} \qquad (\mathbf{1}^H_{n_{\mu,\sigma(t)}} \otimes I_n)X_\mu(t+1) = A(\mathbf{1}^H_{n_{\mu,\sigma(t)}} \otimes I_n)X_\mu(t),$$

which is equivalent to

$$\text{(A.14)} \qquad \bar{x}_\mu(t+1) = A\bar{x}_\mu(t),\ \mu = 1, \ldots, s_{\sigma(t)}.$$

Consider any two different agents i and j. We have two subcases.

Subcase (a). If they belong to one weakly connected component $\mathcal{G}_{\mu,\sigma(t)}$, denote $\delta_{\mu_k}(t) = x_{\mu_k(t)} - \bar{x}_\mu(t), k = 1, \ldots, n_{\mu,\sigma(t)}$, and $\delta_\mu(t) = (\delta^T_{\mu_1}(t), \ldots, \delta^T_{\mu_{n_{\mu,\sigma(t)}}}(t))^T$. Then we have

$$\text{(A.15)} \qquad \begin{aligned}\|x_i(t+1) - x_j(t+1)\| &= \|x_i(t+1) - \bar{x}_\mu(t+1) + \bar{x}_\mu(t+1) - x_j(t+1)\|\\ &\le 2\|\delta_\mu(t+1)\|.\end{aligned}$$

Similarly to case (1), one can prove that for $t \in [t_i, t_{i+1})$,

$$\text{(A.16)} \qquad \delta_\mu(t+1) = (I_{n_{\mu,\sigma(t)}} \otimes A - \mathcal{L}_{\mathcal{G}_{\mu,\sigma(t)}} \otimes BK)\delta_\mu(t).$$

Transform $\mathcal{L}_{\mathcal{G}_{\mu,\sigma(t)}}$ to its Jordan canonical, that is, there exist $\Phi_{\mathcal{G}_{\mu,\sigma(t)}}$, the first row of which is $\mathbf{1}^H_{n_{\mu,\sigma(t)}}$, such that $\Phi_{\mathcal{G}_{\mu,\sigma(t)}}\mathcal{L}_{\mathcal{G}_{\mu,\sigma(t)}}\Phi^{-1}_{\mathcal{G}_{\mu,\sigma(t)}} = diag(\mathbf{0}, J_2^{\mathcal{G}_{\mu,\sigma(t)}}, \ldots, J_{k_{\mu,\sigma(t)}}^{\mathcal{G}_{\mu,\sigma(t)}})$. Since $\mathcal{G}_{\mu,\sigma(t)} \in \mathscr{G}$, we can see that $\|\Phi_{\mathcal{G}_{\mu,\sigma(t)}}\| \le C_1$ and $\|\Phi^{-1}_{\mathcal{G}_{\mu,\sigma(t)}}\| \le C_2$. Let $\tilde{\delta}_\mu(t) = (\Phi_{\mathcal{G}_{\mu,\sigma(t)}} \otimes I_n)\delta_\mu(t)$. Denote the first n elements of $\tilde{\delta}_\mu(t)$ as $\tilde{\delta}_{\mu_1}(t)$ and the others as $\tilde{\delta}_{\mu_2}(t)$. It can be seen that $\tilde{\delta}_{\mu_1}(t) = 0$.

Denote $I_{n_{\mu,\sigma(t)}-1} \otimes A - diag(J_2^{\mathcal{G}_{\mu,\sigma(t)}}, \ldots, J_{k_{\mathcal{G}_{\mu,\sigma(t)}}}^{\mathcal{G}_{\mu,\sigma(t)}}) \otimes BK$ as $A(\mathcal{L}_{\mathcal{G}_{\mu,\sigma(t)}}, K)$; by (A.16) we have

$$\text{(A.17)} \qquad \tilde{\delta}_{\mu_2}(t+1) = A(\mathcal{L}_{\mathcal{G}_{\mu,\sigma(t)}}, K)\tilde{\delta}_{\mu_2}(t).$$

Since $\mathcal{G}_{\mu,\sigma(t)} \in \mathscr{G}$, noting the definition of $\Gamma_1(K)$ and (A.17), we have

$$\|\delta_\mu(t_{i+1})\| \le C_1C_2M_1(\eta_1)\eta_1^{t_{i+1}-t_i}\|\delta_\mu(t_i)\| \le C_1C_2M_1(\eta_1)\|\delta_\mu(t_i)\|.$$

According to [19], we know that $\|\delta_\mu(t)\| \le \|\delta(t)\|$. So by (A.15), if i and j belong to the same weakly connected component in $[t_i, t_{i+1})$, then we have

$$\|x_i(t_{i+1}) - x_j(t_{i+1})\| \le 2C_1C_2M_1(\eta_1)\|\delta(t_i)\|. \tag{A.18}$$

Subcase (b). If i and j belong to two different weakly connected components $\mathcal{G}_{\mu,\sigma(t)}$ and $\mathcal{G}_{\mu',\sigma(t)}$, then we have

$$\begin{aligned}\|x_i(t_{i+1}) - x_j(t_{i+1})\| &= \|x_i(t_{i+1}) - \bar{x}_\mu(t_{i+1}) + \bar{x}_\mu(t_{i+1}) - \bar{x}_{\mu'}(t_{i+1}) \\ &\qquad + \bar{x}_{\mu'}(t_{i+1}) - x_j(t_{i+1})\| \\ &\le \|\delta_\mu(t_{i+1})\| + \|\delta_{\mu'}(t_{i+1})\| + \|\bar{x}_\mu(t_{i+1}) - \bar{x}_{\mu'}(t_{i+1})\|.\end{aligned} \tag{A.19}$$

From the above, we know that

$$\|\delta_\mu(t_{i+1})\| \le C_1C_2M_1(\eta_1)\|\delta(t_i)\|, \|\delta_{\mu'}(t_{i+1})\| \le C_1C_2M_1(\eta_1)\|\delta(t_i)\|. \tag{A.20}$$

By (A.14), we have

$$\|\bar{x}_\mu(t_{i+1}) - \bar{x}_{\mu'}(t_{i+1})\| = \|A^{t_{i+1}-t_i}(\bar{x}_\mu(t_i) - \bar{x}_{\mu'}(t_i))\|. \tag{A.21}$$

According to the definition of $M_0(\eta_0)$, η_0, $R(K,\eta_0,\eta_1)$, and $(A.21)$, we have

$$\begin{aligned}\|\bar{x}_\mu(t_{i+1}) - \bar{x}_{\mu'}(t_{i+1})\| &\le M_0(\eta_0)\eta_0^{t_{i+1}-t_i}\|\bar{x}_\mu(t_i) - \bar{x}_{\mu'}(t_i)\| \\ &\le R(K,\eta_0,\eta_1)\|\bar{x}_\mu(t_i) - \bar{x}_{\mu'}(t_i)\|.\end{aligned}$$

Next we prove that $\|\bar{x}_\mu(t_i) - \bar{x}_{\mu'}(t_i)\| \le 2\|\delta(t_i)\|$. It can be seen that

$$\|\bar{x}_\mu(t_i) - \bar{x}_{\mu'}(t_i)\| \le \|\bar{x}_\mu(t_i) - \bar{x}(t_i)\| + \|\bar{x}(t_i) - \bar{x}_{\mu'}(t_i)\| \tag{A.22}$$

and

$$\begin{aligned}\|\bar{x}_\mu(t_i) - \bar{x}(t_i)\| &= \left\|\sum_{k=1}^{n_\mu}\frac{1}{n_\mu}x_{\mu_k}(t_i) - \bar{x}(t_i)\right\| \\ &= \left\|\sum_{k=1}^{n_\mu}\frac{1}{n_\mu}(x_{\mu_k}(t_i) - \bar{x}(t_i))\right\| \\ &\le \sum_{k=1}^{n_\mu}\frac{1}{n_\mu}\|\delta(t_i)\| = \|\delta(t_i)\|.\end{aligned}$$

With the same method, we know that $\|\bar{x}(t_i) - \bar{x}_{\mu'}(t_i)\| \le \|\delta(t_i)\|$. Then by (A.22), we have $\|\bar{x}_\mu(t_i) - \bar{x}_{\mu'}(t_i)\| \le 2\|\delta(t_i)\|$.

By (A.19) and (A.20), we can see that $\|x_i(t_{i+1}) - x_j(t_{i+1})\| \le 4R(K,\eta_0,\eta_1)\|\delta(t_i)\|$. So, if i and j belong to two different components, then $\|x_i(t_{i+1}) - x_j(t_{i+1})\| \le 4R(K,\eta_0,\eta_1)\|\delta(t_i)\|$. This together with $(A.18)$ and the definition of $R(K,\eta_0,\eta_1)$ leads to that for case (2): for any $i,j \in \{1,\dots,N\}, i \ne j$, there is

$$\|x_i(t_{i+1}) - x_j(t_{i+1})\| \le 4R(K,\eta_0,\eta_1)\|\delta(t_i)\|. \tag{A.23}$$

It can be seen that $\|\delta(t)\| \leq \sqrt{N} \max_{i,j} \|x_i(t) - x_j(t)\|$. Then from (A.23), we have

$$\|\delta(t_{i+1})\| \leq 4\sqrt{N} R(K, \eta_0, \eta_1) \|\delta(t_i)\|. \tag{A.24}$$

Now we prove that $\delta(t)$ tends to 0 exponentially. For any time $t \in \mathbb{N}$, there is an integer i such that $t \in [t_i, t_{i+1})$. For any $k < i$, if $\mathcal{G}_{\sigma(t)}$ is not strongly connected on $[t_k, t_{k+1})$, then by (A.24), we know that

$$\|\delta(t_{k+1})\| \leq 4\sqrt{N} R(K, \eta_0, \eta_1) \|\delta(t_k)\|. \tag{A.25}$$

Denote $[t_{n_k}, t_{n_{k+1}})$, $k = 1, 2, \ldots, n_k \leq i$, as the largest continuous intervals during which $\mathcal{G}_{\sigma(t)}$ is not strongly connected, that is, $\mathcal{G}_{\sigma(t)}$ is not strongly connected in $[t_{n_k}, t_{n_{k+1}})$ but is strongly connected in $[t_{n_k-1}, t_{n_k})$ and $[t_{n_{k+1}}, t_{n_{k+1}+1})$. By Assumption (A5), we have $t_{n_{k+1}} - t_{n_k} \leq T$, so the topology flow switches at most T times in each $[t_{n_k}, t_{n_{k+1}})$. Then by (A.25), and noting that $\eta_1 < 1$, we have

$$\begin{aligned}
\|\delta(t_{n_{k+1}})\| &\leq (4\sqrt{N} R(K, \eta_0, \eta_1))^T \|\delta(t_{n_k})\| \\
&\leq \frac{(4\sqrt{N} R(K, \eta_0, \eta_1))^T}{\eta_1^T} \eta_1^{t_{n_{k+1}} - t_{n_k}} \|\delta(t_{n_k})\|.
\end{aligned}$$

By Assumption (A5), we know that on $[0, t_i)$, there is at most $\frac{t_i}{1+\tau}$ such $[t_{n_k}, t_{n_{k+1}})$. Then according to (A.10) and (A.25), we have

$$\begin{aligned}
\|\delta(t_i)\| &\leq \left(\max\{C_1 C_2 M_1(\eta_1), 1\}\right)^{\frac{t_i}{\tau}} \eta_1^{t_i} \|\delta(0)\| \left(\frac{(4\sqrt{N} R(K, \eta_0, \eta_1))^T}{\eta_1^T}\right)^{\frac{t_i}{\tau+1}} \\
&\leq \left[(\max\{C_1 C_2 M_1(\eta_1), 1\})^{\frac{1}{\tau}} \eta_1^{1 - \frac{T}{\tau+1}} \left(4\sqrt{N} R(K, \eta_0, \eta_1)\right)^{\frac{T}{\tau+1}}\right]^{t_i} \|\delta(0)\| \\
&= (\rho_1(K, \eta_0, \eta_1))^{t_i} \|\delta(0)\|.
\end{aligned} \tag{A.26}$$

If $\mathcal{G}_{\sigma(t)}$ is not strongly connected on $[t_i, t_{i+1})$, then we have $t - t_i \leq t_{i+1} - t_i \leq T$. Noting that $\rho_1(K, \eta_0, \eta_1) < 1$, then by (A.24) and (A.26), we have

$$\|\delta(t)\| \leq \frac{4\sqrt{N} R(K, \eta_0, \eta_1)}{(\rho_1(K, \eta_0, \eta_1))^T} (\rho_1(K, \eta_0, \eta_1))^t \|\delta(0)\|. \tag{A.27}$$

If $\mathcal{G}_{\sigma(t)}$ is strongly connected on $[t_i, t_{i+1})$, then by the definition of $\rho_1(K, \eta_0, \eta_1)$, one can easily see that $\eta_1 < \rho_1(K, \eta_0, \eta_1)$. This together with (A.10) and (A.26) leads to

$$\begin{aligned}
\|\delta(t)\| &\leq C_1 C_2 M_1(\eta_1) \eta_1^{t - t_i} \|\delta(t_i)\| \\
&\leq C_1 C_2 M_1(\eta_1) (\rho_1(K, \eta_0, \eta_1))^t \|\delta(0)\|.
\end{aligned} \tag{A.28}$$

Noting that $C_3(K, \eta_0, \eta_1) = \max\{\frac{4\sqrt{N} R(K, \eta_0, \eta_1)}{(\rho_1(K, \eta_0, \eta_1))^T}, C_1 C_2 M_1(\eta_1)\}$, then according to (A.27) and (A.28), we have

$$\|\delta(t)\| \leq C_3(K, \eta_0, \eta_1) (\rho_1(K, \eta_0, \eta_1))^t \|\delta(0)\|, \ t \in \mathbb{N}. \tag{A.29}$$

So the dynamic network achieves synchronization exponentially and there exist $C_3 = C_3(K, \eta_0, \eta_1)$, $\rho_1 = \rho_1(K, \eta_0, \eta_1)$ such that $\|\delta(t)\| \leq C_3 \rho_1^t \|\delta(0)\|$, $t \in \mathbb{N}$. ∎

LEMMA A.3. *For a dynamic network* $(A, B, \mathcal{G}_{\sigma(t)})$ *satisfying* (A1)–(A5), *if* $\tau > \tau^*$ *with* τ^* *defined as in Lemma* 3.1, *then there exists* $K \in \mathbb{R}^{m\times n}$ *such that the following linear switching system*

$$w(t+1) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}_{\sigma(t)}} \otimes BK)w(t), \tag{A.30}$$

where $w(t) \in \mathbb{R}^{nN}$ *and* $(\mathbf{1}_N^H \otimes I_n)w(0) = 0$, *is exponentially stable. Precisely, there exist* $C_3 \in \mathbb{R}$ *and* $\rho_1 \in \mathbb{R}$, *which are independent of* $w(0)$, *such that* $\|w(t)\| \leq C_3\|w(0)\|\rho_1^t$, $t \in \mathbb{N}$.

Proof. Define a linear multiagent system with agent dynamics being (2.1) and its communication topology flow being $\mathcal{G}_{\sigma(t)}$. Take control protocol (3.1) and let $X(0) = w(0)$. Since $(\mathbf{1}_N^H \otimes I_n)w(0) = 0$, then the initial synchronization errors $\delta(0) = w(0)$. By (A.4), one can easily see that the evolution of the synchronization errors of the defined linear multiagent system is the same as (A.30), so $\delta(t) \equiv w(t)$, $t \in \mathbb{N}$. The rest of the proof is very similar to that of Lemma 3.1 and is omitted here to save space. Noting (A.29), we can see that there exist $C_3 \in \mathbb{R}$ and $\rho_1 \in \mathbb{R}$ such that $\|w(t)\| \leq C_3\|w(0)\|\rho_1^t$, $t \in \mathbb{N}$. ∎

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